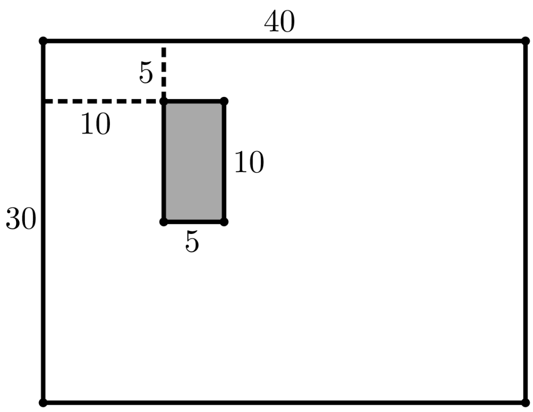
The problems of 5^{th} IGO along with their solutions
Elementary Level

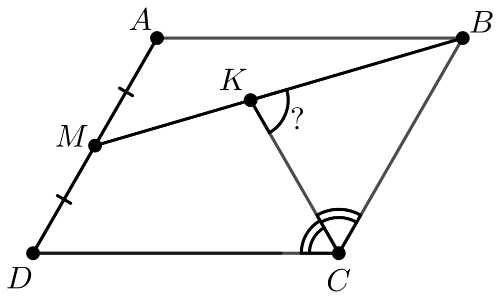
Problems:

1 As shown below, there is a 40×30 paper with a filled 10×5 rectangle inside of it. We want to cut out the filled rectangle from the paper using four straight cuts. Each straight cut is a straight line that divides the paper into two pieces, and we keep the piece containing the filled rectangle. The goal is to minimize the total length of the straight cuts. How to achieve this goal, and what is that minimized length? Show the correct cuts and write the final answer. There is no need to prove the answer.



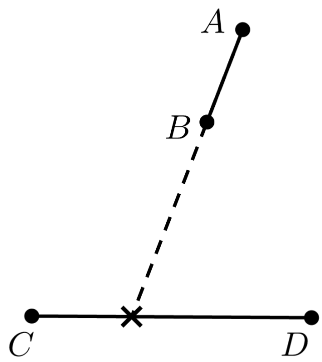
2 Convex hexagon $A_1A_2A_3A_4A_5A_6$ lies in the interior of convex hexagon $B_1B_2B_3B_4B_5B_6$ such that $A_1A_2 \parallel B_1B_2$, $A_2A_3 \parallel B_2B_3$,..., $A_6A_1 \parallel B_6B_1$. Prove that the areas of simple hexagons $A_1B_2A_3B_4A_5B_6$ and $B_1A_2B_3A_4B_5A_6$ are equal. (A simple hexagon is a hexagon which does not intersect itself.)

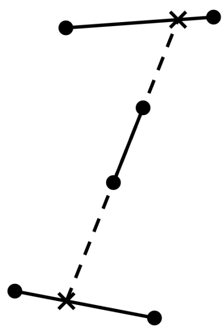
3 In the given figure, $ABCD$ is a parallelogram. We know that $\angle D = 60^\circ$, $AD = 2$ and $AB = \sqrt{3} + 1$. Point M is the midpoint of AD. Segment CK is the angle bisector of C. Find the angle CKB.



4 There are two circles with centers O_1, O_2 lie inside of circle ω and are tangent to it. Chord AB of ω is tangent to these two circles such that they lie on opposite sides of this chord. Prove that $\angle O_1AO_2 + \angle O_1BO_2 > 90°$.

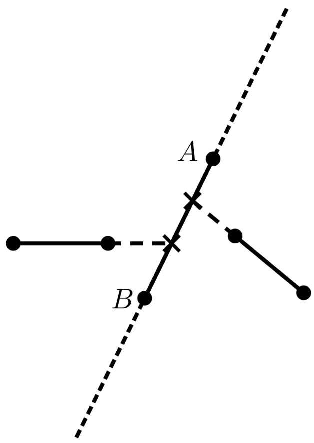
5 There are some segments on the plane such that no two of them intersect each other (even at the ending points). We say segment AB **breaks** segment CD if the extension of AB cuts CD at some point between C and D.





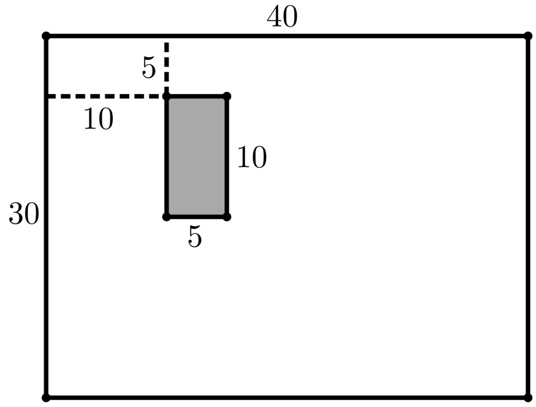
(a) Is it possible that each segment when extended from both ends, breaks exactly one other segment from each way?

(b) A segment is called **surrounded** if from both sides of it, there is exactly one segment that breaks it. (*e.g.* segment AB in the figure.) Is it possible to have all segments to be surrounded?



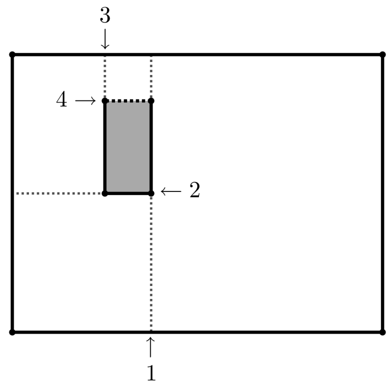
Solutions:

1 As shown below, there is a 40×30 paper with a filled 10×5 rectangle inside of it. We want to cut out the filled rectangle from the paper using four straight cuts. Each straight cut is a straight line that divides the paper into two pieces, and we keep the piece containing the filled rectangle. The goal is to minimize the total length of the straight cuts. How to achieve this goal, and what is that minimized length? Show the correct cuts and write the final answer. There is no need to prove the answer.



Proposed by Morteza Saghafian

Solution. The answer is 65. Here is an example of the solution:



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2 Convex hexagon $A_1A_2A_3A_4A_5A_6$ lies in the interior of convex hexagon $B_1B_2B_3B_4B_5B_6$ such that $A_1A_2 \parallel B_1B_2$, $A_2A_3 \parallel B_2B_3$,..., $A_6A_1 \parallel B_6B_1$. Prove that the areas of simple hexagons $A_1B_2A_3B_4A_5B_6$ and $B_1A_2B_3A_4B_5A_6$ are equal. (A simple hexagon is a hexagon which does not intersect itself.)

Proposed by Mahdi Etesamifard - Hirad Aalipanah

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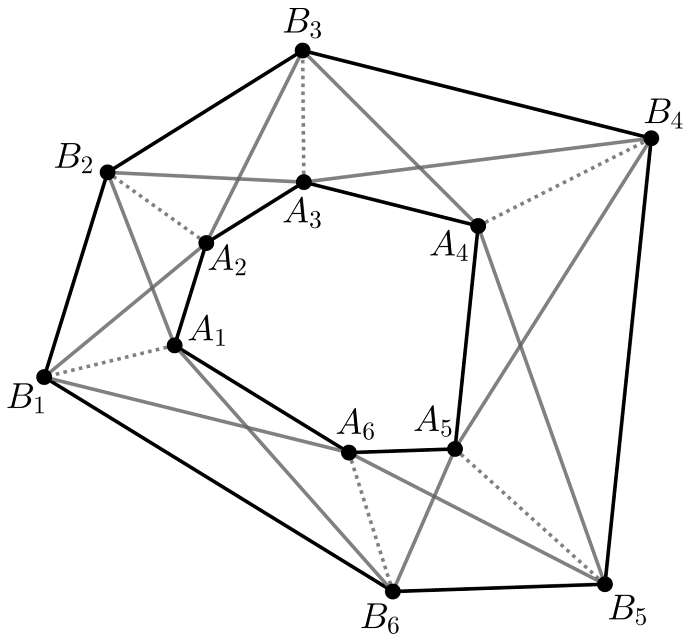
Solution. As you can see, we have divided the area between two polygons into 6 trapezoids. In each trapezoid is it easy to see that the triangles which have the same area (like $B_1A_1A_2$ and $B_2A_1A_2$) each belongs to one of the simple hexagons. Therefore, if we add up their areas and add the common area (the area of $A_1A_2A_3A_4A_5A_6$) to them, we can conclude that the areas of the two simple hexagons are equal.



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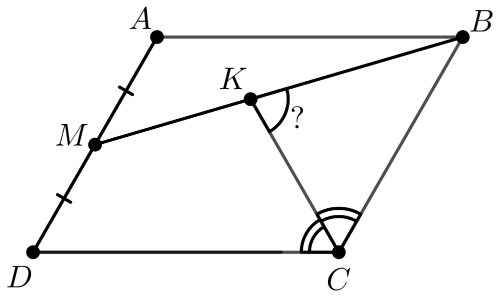
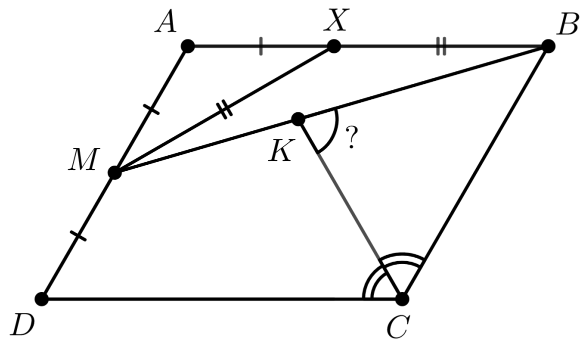
3 In the given figure, $ABCD$ is a parallelogram. We know that $\angle D = 60°$, $AD = 2$ and $AB = \sqrt{3} + 1$. Point M is the midpoint of AD. Segment CK is the angle bisector of C. Find the angle CKB.



Proposed by Mahdi Etesamifard

Solution 1. Let X be a point on AB such that $AX = 1$ and $XB = \sqrt{3}$. We know that $\angle MAX = 120°$. Therefore by Pythagoras theorem we know that $MX = \sqrt{3}$. So we have $\angle MBX = 15°$ and $\angle CBK = 45°$. Hence, $\angle CKB = 180° - 60° - 45° = 75°$.



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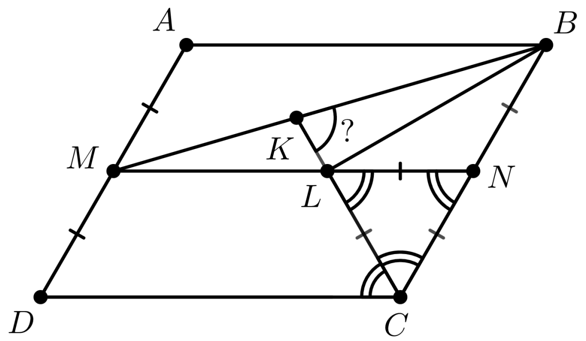
Solution 2. Let N be the midpoint of side BC. MN intersects CK at L. It's clear that the triangle CNL is equilateral. Therefore, we have $LN = CN = NB$. So, BCL is a right-angled triangle. Because of Pythagoras's theorem we have $BL = \sqrt{3}$. On the other hand, we have $ML = \sqrt{3}$ and $\angle BLN = 30°$. Because of that, we have $\angle LBM = 15°$ and so we have $\angle CBK = 30° + 15° = 45°$. Hence, $\angle CKB = 180° - 60° - 45° = 75°$.



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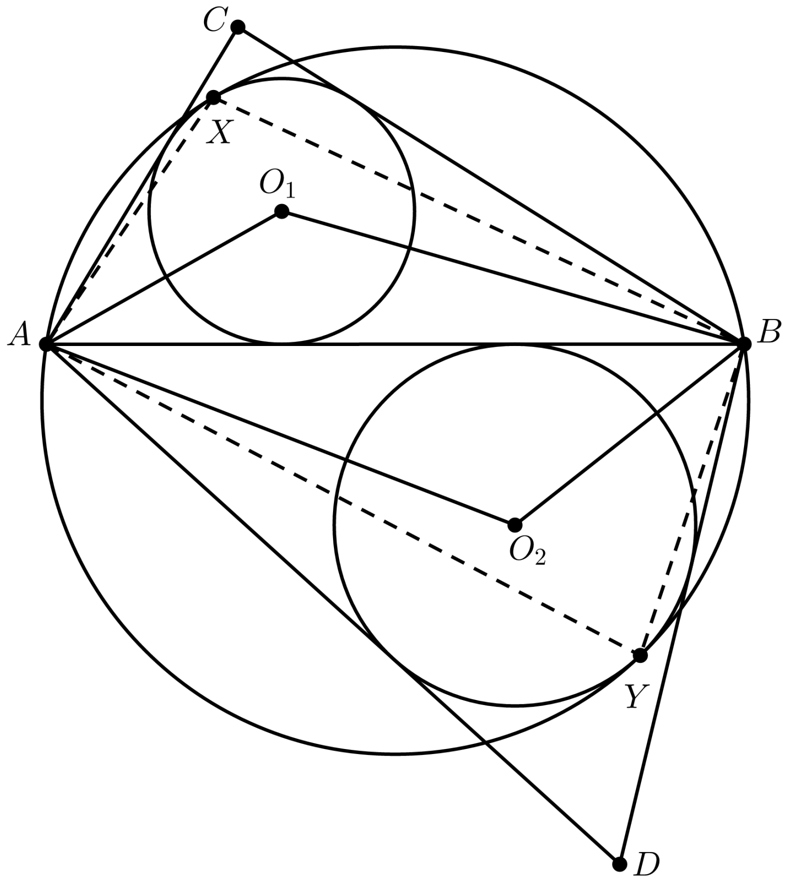
4 There are two circles with centers O_1, O_2 lie inside of circle ω and are tangent to it. Chord AB of ω is tangent to these two circles such that they lie on opposite sides of this chord. Prove that $\angle O_1AO_2 + \angle O_1BO_2 > 90^\circ$.

Proposed by Iman Maghsoudi

Solution. Let AC, BC be tangents from A, B to the circle with center O_1 and AD, BD be tangents from A, B to the circle with center O_2. It's enough to show that $\angle CAD + \angle CBD > 180^\circ$. Or to show that $\angle ACB + \angle ADB < 180^\circ$.

We know that C, D lie on the outside of circle ω. Therefore, we can always say that $\angle ACB < \angle AXB$ and $\angle ADB < \angle AYB$ because of the exterior angles. But we know that $\angle AXB + \angle AYB = 180^\circ$. Hense, we can conclude that $\angle ACB + \angle ADB < 180^\circ$ and the statement is proven.



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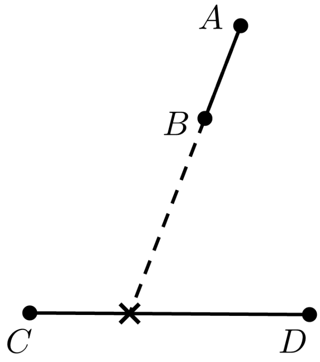
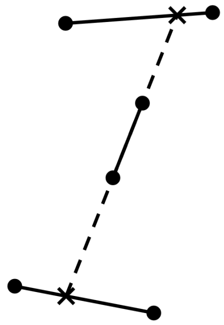
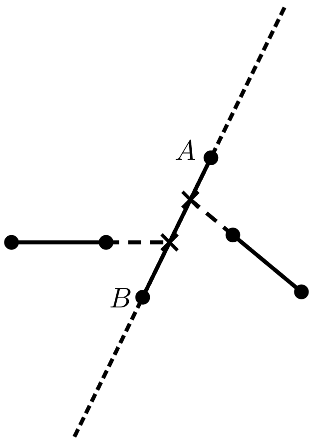
5 There are some segments on the plane such that no two of them intersect each other (even at the ending points). We say segment AB **breaks** segment CD if the extension of AB cuts CD at some point between C and D.





(a) Is it possible that each segment when extended from both ends, breaks exactly one other segment from each way?

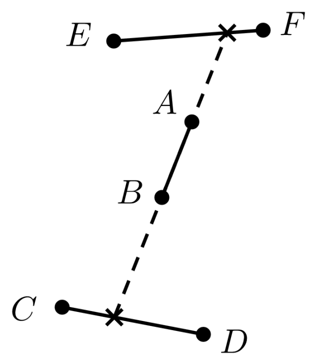


(b) A segment is called **surrounded** if from both sides of it, there is exactly one segment that breaks it.
(*e.g.* segment AB in the figure.) Is it possible to have all segments to be surrounded?

Proposed by Morteza Saghafian

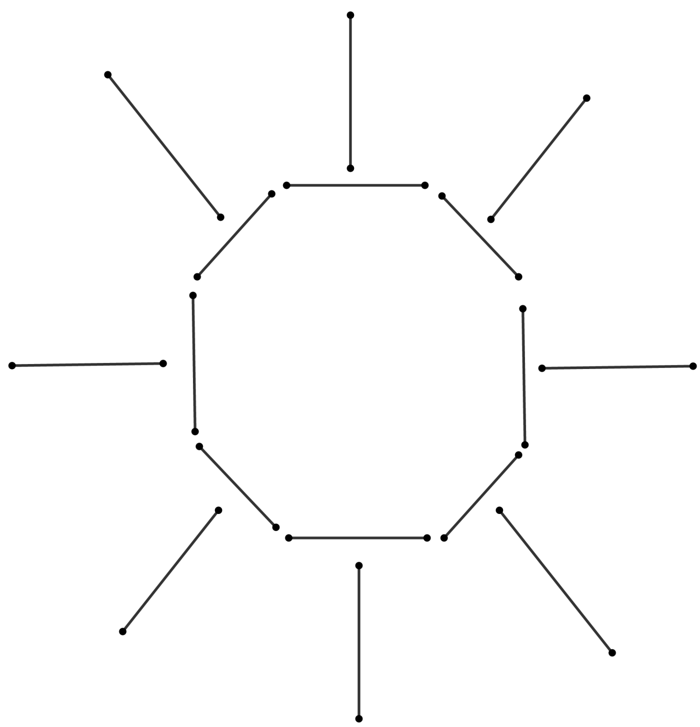
Solution.

(a) No. Consider the convex hull of the endpoints of these segments. Let A be a vertex of the convex hull, where AB is one of the segments.



We know that there exist segments CD, EF as in the figure. So A lies inside the convex hull of C, D, E, F and therefore it cannot be a vertex of the main convex hull. Contradiction! $\square$

(b) Yes. The figure below shows that it is possible for all segments to be surrounded.



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